

JORDAN ALGEBRAS AND ORTHOGONAL POLYNOMIALS

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ABSTRACT. We illustrate how Jordan algebras can provide a framework for the interpretation of certain classes of orthogonal polynomials. The big -1 Jacobi polynomials are eigenfunctions of a first order operator of Dunkl type. We consider an algebra that has this operator (up to constants) as one of its three generators and whose defining relations are given in terms of anticommutators. It is a special case of the Askey-Wilson algebra $AW(3)$. We show how the structure and recurrence relations of the big -1 Jacobi polynomials are obtained from the representations of this algebra. We also present ladder operators for these polynomials and point out that the big -1 Jacobi polynomials satisfy the Hahn property with respect to a generalized Dunkl operator.

1. INTRODUCTION

The relation between Lie groups and special functions is a celebrated one and has been the object of many books [25], [17], [18]. Special functions encode symmetries described by algebras. Over the years, tremendous cross-fertilization has occurred within these areas. The identification of new algebraic structures say like quantum groups, double affine Hecke algebras, polynomial algebras etc, in connection with new manifestations of symmetry generally permit to advance the theory of special functions. Conversely, the discovery of new special functions has often revealed new algebraic tools. This paper relates to the latter.

We have investigated recently the classes of orthogonal polynomials that are eigenfunctions of Dunkl operators, that is of differential-difference operators involving the reflection operator. These studies have allowed to discover new families of classical polynomials. They have also brought to light interesting algebras defined through Jordan products, that is anticommutators.

The big -1 Jacobi polynomials that we define next are the most general eigenpolynomials of first order Dunkl operators. They have an associated Jordan algebra which is readily obtained from the operator of which they are the eigenfunctions. One purpose of this paper is to illustrate how the representations of this algebra largely characterize the corresponding polynomials; another is to provide the ladder operators for the big -1 Jacobi polynomials.

Let us point out that the little -1 Jacobi polynomials have already found applications in physical contexts. These polynomials are obtained from the big -1 Jacobi polynomials by setting one parameter equal to zero. They have been seen, in particular, to arise in the wavefunctions of a supersymmetric Scarf Hamiltonian with reflections [19]. They have shown up furthermore in the angular part of the separated solutions of the Schrödinger equations associated to an infinite family of

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quantum superintegrable models in the plane [20]. This already casts the Jordan algebra associated to the little -1 Jacobi polynomials as the dynamical algebra of these physical systems.

2. BIG -1 JACOBI POLYNOMIALS

The big -1 Jacobi polynomials $P_n^{(\alpha,\beta)}(x; c)$ are the eigensolutions of

$$(2.1) \quad L^{(\alpha,\beta,c)} P_n^{(\alpha,\beta)}(x; c) = \lambda_n P_n^{(\alpha,\beta)}(x; c),$$

where the operator

$$(2.2) \quad L^{(\alpha,\beta,c)} = g_0(x)(R - I) + g_1(x)\partial_x R$$

with

$$(2.3) \quad g_0(x) = \frac{(\alpha + \beta + 1)x^2 + (c\alpha - \beta)x + c}{x^2}, \quad g_1(x) = \frac{2(x-1)(x+c)}{x},$$

I is the identity operator and R the reflection operator $Rf(x) = f(-x)$.

The eigenvalues of $L^{(\alpha,\beta,c)}$ are

$$(2.4) \quad \lambda_n = \begin{cases} 2n, & n \text{ even} \\ -2(\alpha + \beta + n + 1), & n \text{ odd} \end{cases}.$$

As shown in [27] the operator (2.2) is the most general operator of the first order in ∂_x and involving R that has orthogonal polynomials as eigenfunctions. The big -1 Jacobi polynomials depend on 3 parameters α, β, c . When $c = 0$ they become the little -1 Jacobi polynomials [26].

When $\alpha > -1$, $\beta > -1$ and $0 < c < 1$, the big -1 Jacobi polynomials are orthogonal on the union of two symmetric intervals $[-1, -c]$ and $[c, 1]$ of the real axis:

$$(2.5) \quad \int_{-1}^{-c} P_n^{(\alpha,\beta)}(x; c) P_m^{(\alpha,\beta)}(x; c) w(x) dx + \int_c^1 P_n^{(\alpha,\beta)}(x; c) P_m^{(\alpha,\beta)}(x; c) w(x) dx = h_n \delta_{nm}$$

with positive normalization constants h_n . The weight function is

$$(2.6) \quad w(x) = \frac{x}{|x|} (x+1)(x-c)(1-x^2)^{(\alpha-1)/2} (x^2-c^2)^{(\beta-1)/2}.$$

In [26] and [28], we studied the properties of the big and little -1 Jacobi polynomials directly from the limit $q \rightarrow -1$ of the corresponding big and little q -Jacobi polynomials.

We here wish to derive the structure relations and the recurrence coefficients of these polynomials by starting from the eigenvalue equation (2.1). The main tool in this analysis will be a linear Jordan algebra with 3 generators which is a special case of the Askey-Wilson algebra $AW(3)$ [29] for $q = -1$. This algebra was already presented in [26], [28]; we illustrate below its usefulness.

3. JORDAN ALGEBRA FOR BIG -1 JACOBI POLYNOMIALS AND INTERTWINING OPERATORS

Introduce the operators

$$(3.1) \quad X = \frac{1}{2} \left(L^{(\alpha,\beta,c)} + \alpha + \beta + 1 \right), \quad Y = 2x, \quad Z = -\frac{2}{x} (c + (x-1)(x+c)R).$$

In [28] it was shown that these operators are closed in frames of the Jordan algebra and that they satisfy the linear anticommutation relations

$$(3.2) \quad \{X, Y\} = Z + \omega_3, \quad \{Y, Z\} = \omega_1, \quad \{Z, X\} = Y + \omega_2,$$

where

$$\omega_1 = -8c, \quad \omega_2 = 2(\alpha - \beta c), \quad \omega_3 = 2(\beta - \alpha c),$$

with $\{A, B\} = AB + BA$ denoting as usual the anticommutator of A and B . (Note that our definition of the operators X, Y slightly differs from that of [28].)

Observe that the first relation in (3.2) can be taken to be the definition of Z as the anticommutator (up to additive and multiplicative constants) of the operator defining the eigenvalue problem and the operator multiplication by x .

The Casimir operator of the algebra defined by (3.2) is

$$(3.3) \quad Q = Z^2 + Y^2.$$

In our realization the Casimir operator takes the constant value $Q = 4(c^2 + 1)$.

The algebra defined in (3.2) can be considered as the limit $q \rightarrow -1$ of the Askey-Wilson algebra $AW(3)$ [29]. It also belongs to the class of Jordan algebras: the anticommutators of any pair of generators are expressed in terms of the generators. In the present case we have a linear Jordan algebra with 3 generators X, Y, Z .

Consider the canonical polynomial basis

$$(3.4) \quad \Phi_n(x) = \begin{cases} (x^2 - c^2)^{\frac{n}{2}}, & n \text{ even} \\ (x + c)(x^2 - c^2)^{\frac{n-1}{2}}, & n \text{ odd} \end{cases}.$$

The operator $L^{(\alpha, \beta, c)}$ is 2-diagonal and lower-triangular in this basis:

$$(3.5) \quad L^{(\alpha, \beta, c)} \Phi_n(x) = \lambda_n \Phi_n(x) + \eta_n \Phi_{n-1}(x),$$

where

$$(3.6) \quad \eta_n = \begin{cases} 2(c-1)n, & n \text{ even} \\ 2(c+1)(\beta+n), & n \text{ odd} \end{cases}.$$

Clearly, the operator Y is 2-diagonal and upper-triangular in the basis $\Phi_n(x)$. The existence of the basis $\Phi_n(x)$ with such properties is an essential part of Terwilliger's approach to Leonard pairs [22]. From (3.5) it is possible to obtain explicit expressions for the big -1 Jacobi polynomials in terms of hypergeometric functions [28].

An important property of the algebra (3.2) is that it possesses simple intertwining operators J_{\pm} . We define these operators by the formulas

$$(3.7) \quad J_+ = (Y + Z)(X - 1/2) - \frac{\omega_2 + \omega_3}{2}$$

and

$$(3.8) \quad J_- = (Y - Z)(X + 1/2) + \frac{\omega_2 - \omega_3}{2}.$$

From the defining relations (3.2), we find that the operators J_{\pm} satisfy the anti-commutation relations

$$(3.9) \quad \{X, J_+\} = J_+, \quad \{X, J_-\} = -J_-.$$

It is seen readily that both J_+^2 and J_-^2 commute with the operator X :

$$(3.10) \quad [X, J_+^2] = [X, J_-^2] = 0.$$

Let $\psi_n(x)$ be any eigenfunction of the operator X :

$$X\psi_n(x) = \mu_n\psi_n(x),$$

where

$$(3.11) \quad \mu_n = \frac{1}{2}(\lambda_n + \alpha + \beta + 1) = (-1)^n \left(n + \frac{\alpha + \beta + 1}{2} \right).$$

From (3.9) it is immediate to see that the function $\tilde{\psi}_n(x) = J_+\psi_n$ is also an eigenfunction of the operator X with the eigenvalue $\tilde{\mu}_n = 1 - \mu_n$. In view of (3.11) $\tilde{\mu}_n = \mu_{n-1}$ if n is even and $\tilde{\mu}_n = \mu_{n+1}$ if n is odd.

Similar observations are made when J_- replaces J_+ . The function $J_-\psi_n$ is again an eigenfunction of the operator X with eigenvalue $-1 - \mu_n$ which is μ_{n+1} for even n and μ_{n-1} for odd n .

It is also clear that the operators J_\pm transform polynomials into polynomials (but not of the same degree). This means that the operator J_+ transforms big -1 Jacobi polynomials of degree n into big -1 Jacobi polynomials of degree $n \mp 1$. Similarly, the operator J_- transforms big -1 Jacobi polynomials of degree n into big -1 Jacobi polynomials of degree $n \pm 1$. (The upper sign corresponds to even n , the lower sign to odd n .)

Taking into account the leading coefficients, we arrive at the following formulas for the action of J_\pm on big -1 Jacobi polynomials

$$(3.12) \quad J_+P_n(x) = \begin{cases} \frac{2(c-1)^2n(\alpha+\beta+n)}{\alpha+\beta+2n}P_{n-1}(x), & \text{if } n \text{ even} \\ -2(\alpha + \beta + 2(n+1))P_{n+1}(x), & \text{if } n \text{ odd} \end{cases}$$

and similarly

$$(3.13) \quad J_-P_n(x) = \begin{cases} 2(\alpha + \beta + 2(n+1))P_{n+1}(x), & \text{if } n \text{ even} \\ -\frac{2(c+1)^2(\alpha+n)(\beta+n)}{\alpha+\beta+2n}P_{n-1}(x), & \text{if } n \text{ odd} \end{cases}.$$

Note that formulas (3.12) and (3.13) show that the operators J_\pm are block-diagonal in the basis of the polynomials $P_n(x)$ with any block a 2×2 matrix. They basically provide a representation of the algebra (3.2).

4. STRUCTURE RELATIONS FOR THE BIG -1 JACOBI POLYNOMIALS

We shall now derive the structure relations for the big -1 Jacobi polynomials using the formulas obtained in the previous section.

Define the operator $U_n^{(1)}$, $n = 0, 1, 2, \dots$ as

$$(4.1) \quad U_n^{(1)} = \begin{cases} J_+, & \text{if } n \text{ even} \\ J_-, & \text{if } n \text{ odd} \end{cases}.$$

We then have

$$(4.2) \quad U_n^{(1)}P_n(x) = \epsilon_n^{(1)}P_{n-1}(x),$$

where

$$(4.3) \quad \epsilon_n^{(1)} = \begin{cases} \frac{2(c-1)^2n(\alpha+\beta+n)}{\alpha+\beta+2n}, & \text{if } n \text{ even} \\ -\frac{2(c+1)^2(\alpha+n)(\beta+n)}{\alpha+\beta+2n}, & \text{if } n \text{ odd} \end{cases}.$$

Similarly, define the operator $U_n^{(2)}$, $n = 0, 1, 2, \dots$ as

$$(4.4) \quad U_n^{(2)} = \begin{cases} J_-, & \text{if } n \text{ even} \\ J_+, & \text{if } n \text{ odd} \end{cases}.$$

This entails

$$(4.5) \quad U_n^{(2)} P_n(x) = \epsilon_n^{(2)} P_{n+1}(x),$$

where

$$(4.6) \quad \epsilon_n^{(2)} = \begin{cases} -2(\alpha + \beta + 2(n+1)), & \text{if } n \text{ even} \\ 2(\alpha + \beta + 2(n+1)), & \text{if } n \text{ odd} \end{cases}.$$

It is seen that the operator $U_n^{(1)}$ plays the role of a lowering operator, while the operator $U_n^{(2)}$ acts as a raising operator for the polynomials $P_n(x)$. Relations such as (4.2) and (4.5) are called structure relations: the operators $U_n^{(1,2)}$ increment by ± 1 the degree of the big -1 Jacobi polynomials without altering the parameters α, β, c . Note however, that the operators $U_n^{(1,2)}$ depend on n . Here this dependence is "minimal": even and odd sequences of these operators do not depend on n .

These structure relations proved essential in demonstrating the superintegrability of the two-dimensional infinite family of quantum systems with Hamiltonians [20]:

$$(4.7) \quad H_k(r, \theta; \omega, \alpha, \beta) = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_\theta^2 + \omega^2 r + \frac{\alpha k^2}{2} \left(\frac{\frac{\alpha}{2} - \cos k\theta}{r^2 \sin^2 k\theta} R \right) + \frac{\beta k^2}{2} \left(\frac{\frac{\beta}{2} - \sin k\theta}{r^2 \cos^2 k\theta} R \right),$$

where R is the reflection operators with respect to θ

$$(4.8) \quad Rf(\theta) = f(-\theta).$$

The following restrictions on the real parameters α, β , and k : $\alpha > -1, \beta > -1, k \neq 0$ are imposed with $r \in [0, \infty)$ and $-\frac{\pi}{2k} \leq \theta \leq \frac{\pi}{2k}$.

The angular part of H (in the variable θ) provides also a supersymmetrization with reflections of the Scarf system in one dimension [19]. With the help of the above structure relations, following the recurrence approach of [13], the constants of motion of (4.7) could be constructed [20], thereby making the superintegrability manifest. Furthermore, it was possible to determine the polynomial algebra realized by these conserved quantities.

5. RECURRENCE RELATIONS

Using the operators J_\pm it is easy to derive the 3-term recurrence relations of the polynomials $P_n(x)$.

To that end we consider the operator

$$(5.1) \quad V = J_+(X + 1/2) + J_-(X - 1/2).$$

Recall that X is diagonal in the polynomials basis $P_n(x)$: $LP_n(x) = \mu_n P_n(x)$, while J_\pm are 2-diagonal in the same basis. Using formulas (3.12) and (3.13) we have on the one hand,

$$(5.2) \quad VP_n(x) = \begin{cases} \xi_n^{(-)} P_{n-1}(x) + \xi_n^{(+)} P_{n+1}(x), & \text{if } n \text{ even} \\ \eta_n^{(-)} P_{n-1}(x) + \eta_n^{(+)} P_{n+1}(x) & \text{if } n \text{ odd} \end{cases},$$

where

$$\xi_n^{(-)} = \frac{2(c-1)^2 n(\alpha + \beta + n)(\mu_n + 1/2)}{\alpha + \beta + 2n}, \quad \xi_n^{(+)} = 2(\mu_n - 1/2)(\alpha + \beta + 2(n+1))$$

and

$$\eta_n^{(-)} = -2(\mu_n - 1/2)(\alpha + \beta + 2(n+1)), \quad \eta_n^{(+)} = -\frac{2(c+1)^2(\alpha + n)(\beta + n)(\mu_n + 1/2)}{\alpha + \beta + 2n}.$$

(Recall that μ_n is the eigenvalue of the operator X given by (3.11).)

On the other hand, (3.7) and (3.8) can be used to present the operator V of (5.1) in the form

$$(5.3) \quad V = 2Y(X^2 - 1/4) - \omega_3 X - \omega_2/2.$$

The operator Y coincides with the operator multiplication by $2x$, so that in the polynomial basis $P_n(x)$ we have $YP_n(x) = 2xP_n(x)$. Hence from (5.3) we have

$$(5.4) \quad VP_n(x) = (2(\mu_n^2 - 1/4)x - \omega_3\mu_n - \omega_2/2)P_n(x).$$

Comparing (5.2) and (5.4), we arrive at the 3-term recurrence relation for the polynomials $P_n(x)$:

$$(5.5) \quad xP_n(x) = P_{n+1}(x) + b_nP_n(x) + u_nP_{n-1}(x),$$

where the recurrence coefficients are

$$(5.6) \quad u_n = \begin{cases} \frac{(1-c)^2 n(\alpha + \beta + n)}{(\alpha + \beta + 2n)^2}, & n \text{ even} \\ \frac{(1+c)^2(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n)^2}, & n \text{ odd} \end{cases}$$

and

$$(5.7) \quad b_n = \begin{cases} -c + \frac{(c-1)n}{\alpha + \beta + 2n} + \frac{(1+c)(\beta + n + 1)}{\alpha + \beta + 2n + 2}, & n \text{ even} \\ c + \frac{(1-c)(n+1)}{\alpha + \beta + 2n + 2} - \frac{(c+1)(\beta + n)}{\alpha + \beta + 2n}, & n \text{ odd} \end{cases}.$$

The expressions (5.6), (5.7) coincide with the recurrence coefficients of the big -1 Jacobi polynomials found in [28] by a direct $q \rightarrow -1$ limit of the big q -Jacobi polynomials counterparts.

6. LOWERING AND RAISING OPERATORS FOR THE BIG -1 JACOBI POLYNOMIALS

Consider the operator

$$(6.1) \quad \mathfrak{D} = A(x)(I - R) + B(x)\partial_x + C(x)\partial_x R,$$

where

$$(6.2) \quad A(x) = \frac{c^2}{x^3} - \frac{c(c-1)}{2x^2} + \frac{\beta(c+1)}{2x}, \quad B(x) = \frac{x^2 - c^2}{x^2}, \quad C(x) = \frac{c(x+c)(1-x)}{x^2}.$$

It is easily verified that the operator \mathfrak{D} transforms any polynomial of degree n into a polynomial of degree $n-1$. Moreover, on the basis $\Phi_n(x)$ defined by (3.4) this operator acts simply as follows:

$$(6.3) \quad \mathfrak{D}\Phi_n(x) = \nu_n \Phi_{n-1}(x),$$

where

$$(6.4) \quad \nu_n = \begin{cases} (1-c)n, & n \text{ even} \\ (c+1)(\beta + n), & n \text{ odd.} \end{cases}$$

When $c = 0$ the operator \mathfrak{D} becomes the ordinary Dunkl operator

$$(6.5) \quad \mathfrak{D}|_{c=0} = \partial_x + \frac{\beta}{2x}(I - R).$$

Thus the operator \mathfrak{D} can be considered as a natural generalization of the Dunkl operator with respect to the basis $\Phi_n(x)$.

The operator \mathfrak{D} satisfies an important intertwining property

$$(6.6) \quad L^{(\alpha+2,\beta,c)}\mathfrak{D} + \mathfrak{D}L^{(\alpha,\beta,c)} + 2(\alpha + \beta + 2)\mathfrak{D} = 0.$$

Relation (6.6) can be verified by direct calculations.

From (6.6) follows that for ψ_n an eigenfunction of the operator $L^{(\alpha,\beta,c)}$

$$L^{(\alpha,\beta,c)}\psi_n(x) = \lambda_n\psi_n(x),$$

the function $\tilde{\psi}_n(x) = \mathfrak{D}\psi_n(x)$ is an eigenfunction of the operator $L^{(\alpha+2,\beta,c)}$:

$$(6.7) \quad L^{(\alpha+2,\beta,c)}\tilde{\psi}_n(x) = \tilde{\lambda}_n\tilde{\psi}_n(x)$$

with

$$(6.8) \quad \tilde{\lambda}_n = -\lambda_n - 2(\alpha + \beta + 2).$$

We note also that the operator \mathfrak{D} transforms polynomials of degree n into polynomials of degree $n - 1$. Hence the operator \mathfrak{D} transforms the big -1 Jacobi polynomials $P_n^{(\alpha,\beta)}(x; c)$ (which are the unique polynomial eigenfunctions of the operator $L^{(\alpha,\beta,c)}$) into the polynomials $P_{n-1}^{(\alpha+2,\beta)}(x; c)$ (which are the unique polynomial eigenfunctions of the operator $L^{(\alpha+2,\beta,c)}$):

$$(6.9) \quad \mathfrak{D}P_n^{(\alpha,\beta)}(x; c) = \nu_n P_{n-1}^{(\alpha+2,\beta)}(x; c),$$

where ν_n is given by (6.4).

The operator \mathfrak{D} is thus a lowering operator for the big -1 Jacobi polynomials. Moreover, we proved that the big -1 Jacobi polynomials possess the Hahn property: namely that there exists an operator \mathfrak{D} such that its application to the polynomials $P_n^{(\alpha,\beta)}(x; c)$ gives another set of orthogonal polynomials $P_n^{(\alpha+2,\beta)}(x; c)$.

In the case of little -1 Jacobi polynomials (i.e. when $c = 0$), the Hahn property had been proven in [26]

It is well known [7] that the classical orthogonal polynomials are completely characterized by the Hahn property with respect to the ordinary derivative operator ∂_x . The little -1 Jacobi polynomials satisfy their Hahn property with respect to the Dunkl operator (6.5). We see that the big -1 Jacobi polynomials satisfy the Hahn property with respect to a generalized Dunkl operator \mathfrak{D} . An interesting open question is to characterize all orthogonal polynomials that have the Hahn property with respect to generalized Dunkl operators (which contain the operators ∂_x and R).

Note also that the action of the operator \mathfrak{D} is equivalent to the application of two Christoffel transforms to the big -1 Jacobi polynomials.

Indeed, it is well known (see, e.g. [30]) that the Christoffel transform with parameter a

$$(6.10) \quad \tilde{P}_n(x) = \frac{P_{n+1}(x) - \frac{P_{n+1}(a)}{P_n(a)}P_n(x)}{x - a}$$

is equivalent to the multiplication of their weight function by a linear factor: $\tilde{w}(x) = (x-a)w(x)$. A multiple Christoffel transform is hence equivalent to the multiplication of the weight function by a polynomial: $\tilde{w}(x) = (x-a_1)(x-a_2)\dots(x-a_N)w(x)$.

It is obvious from (2.6) that the weight function $w^{(\alpha+2,\beta,c)}(x)$ corresponding to the big -1 Jacobi polynomials $P_n^{(\alpha+2,\beta)}(x;c)$ and the weight function $w^{(\alpha,\beta,c)}(x)$ corresponding to the big -1 Jacobi polynomials $P_n^{(\alpha,\beta)}(x;c)$, differ by a factor: $1-x^2 = (1-x)(1+x)$

$$(6.11) \quad w^{(\alpha+2,\beta,c)}(x) = (1-x^2)w^{(\alpha,\beta,c)}(x).$$

Hence the big -1 Jacobi polynomials $\mathfrak{D}P_n^{(\alpha,\beta)}(x;c)$ are obtained from the polynomials $P_n^{(\alpha,\beta)}(x;c)$ by two successive Christoffel transforms at the points $a_1 = 1$ and $a_2 = -1$. It is interesting to note that a similar property is valid for the ordinary Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Indeed, Geronimus showed [7] that the polynomials $\partial_x P_{n+1}^{(\alpha,\beta)}(x)$ are obtained from the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ by two successive Christoffel transforms at the points $a_1 = 1$ and $a_2 = -1$.

In the same manner one can construct a raising operator for the big -1 Jacobi polynomials.

Introduce the operator

$$(6.12) \quad \mathfrak{R} = S_1(x)I + S_2(x)R + T_1(x)\partial_x + T_2(x)\partial_x R,$$

where

$$(6.13) \quad S_1(x) = \frac{(\beta c - \beta - 2\alpha)x - (c+2)(c-1) + \beta - 2c^2 + 2c^2\alpha - \beta c}{x} - \frac{c(c-1)}{x^2} + \frac{2c^2}{x^3},$$

$$(6.14) \quad S_2(x) = \frac{(\beta c - \beta + 2c\alpha)x + (c-1)(2c\alpha - c - 2\beta) - \beta - 2c^2\alpha + \beta c + 2c^2}{x} + \frac{c(c-1)}{x^2} - \frac{2c^2}{x^3}$$

and

$$(6.15) \quad T_1(x) = \frac{2(x^2-1)(c^2-x^2)}{x^2}, \quad T_2(x) = \frac{2c(1+x)(x-1)^2(x+c)}{x^2}.$$

It is easily seen that the operator \mathfrak{R} transforms any polynomial of degree n into a polynomial of degree $n+1$.

Moreover, there is an intertwining property

$$(6.16) \quad L^{(\alpha-2,\beta,c)}\mathfrak{R} + \mathfrak{R}L^{(\alpha,\beta,c)} + 2(\alpha+\beta)\mathfrak{R} = 0.$$

Hence the operator \mathfrak{R} maps the polynomials $P_n^{(\alpha,\beta)}(x;c)$ to the polynomials $P_{n+1}^{(\alpha-2,\beta)}(x;c)$:

$$(6.17) \quad \mathfrak{R}P_n^{(\alpha,\beta)}(x;c) = \kappa_n P_{n+1}^{(\alpha-2,\beta)}(x;c),$$

where

$$(6.18) \quad \kappa_n = \begin{cases} 2(c-1)(\alpha+\beta+n), & n \text{ even} \\ -2(c+1)(\alpha+n), & n \text{ odd.} \end{cases}$$

When $c = 0$, the expression for the raising operator becomes more simple:

$$(6.19) \quad \mathfrak{R} = 2(1-x^2)\partial_x R - \frac{\beta(x-1)^2}{x}R + \left(2 + \frac{\beta}{x} - (\beta+2\alpha)x\right)I.$$

Expression (6.19) gives the raising operator for the little -1 Jacobi polynomials found in [26].

We mentioned in the introduction that the little -1 Jacobi polynomials form the wavefunctions of the supersymmetric Scarf Hamiltonian (which is the angular part of (4.7)). As was pointed out in [19], in view of their intertwining relations, for $c = 0$, the operators (6.5) and (6.19) allow to connect eigenfunctions of such extended Scarf Hamiltonians with different parameters.

7. CONCLUSIONS

We have shown that algebras, whose defining relations are given in terms of Jordan products, can be naturally associated to certain families of orthogonal polynomials. We have used for purposes of illustration the example of the big and little -1 polynomials. We made the case of the usefulness of such Jordan algebras in deriving structural properties of the corresponding polynomials. We noted that there are some physical systems for which the algebra we have studied is dynamical. We developed in [24] the theory of the Bannai-Ito polynomials where again a Jordan algebra was key. We trust in concluding that such structures will continue to appear in various guises (physical and mathematical) and will warrant further analysis.

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